Theorem 1 (Cauchy-Schwarz inequality). *Given real numbers a, b, the following holds*

$$2ab \leq a^2 + b^2.$$

Proof.

$$0 \leq |a-b|^2 = a^2 + b^2 - 2ab.$$

□

Theorem 2 (AM-GM). *Given positive real numbers $a_1, \cdots, a_n$, the following holds*

$$\Big(\prod_{i=1}^{n} a_i\Big)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{i=1}^{n} a_i.$$

Proof. We have the desired result for $n = 2$ in the previous theorem. Assume that the inequality holds for $n = 2^k$. Then, given positive real numbers $a_1, \cdots, a_{2^{k+1}}$ we have

$$\Big(\prod_{i=1}^{2^k} a_i\Big)^{\frac{1}{2^k}} \leq \frac{1}{2^k}\sum_{i=1}^{2^k} a_i, \qquad \Big(\prod_{k=2^k+1}^{2^{k+1}} a_j\Big)^{\frac{1}{2^{k+1}}} \leq \frac{1}{2^k}\sum_{j=2^k+1}^{2^{k+1}} a_j.$$

In addition, the theorem above yields

$$2\Big(\prod_{i=1}^{2^{k+1}} a_i\Big)^{\frac{1}{2^{k+1}}} = 2\Big(\prod_{i=1}^{2^k} a_i\Big)^{\frac{1}{2^{k+1}}}\Big(\prod_{j=2^k+1}^{2^{k+1}} a_j\Big)^{\frac{1}{2^{k+1}}} \leq \Big(\prod_{i=1}^{2^k} a_i\Big)^{\frac{1}{2^k}} + \Big(\prod_{i=1}^{2^k} a_i\Big)^{\frac{1}{2^k}}.$$

We combine the above inequalities so that we have

$$2\Big(\prod_{i=1}^{2^{k+1}} a_i\Big)^{\frac{1}{2^{k+1}}} \leq \frac{1}{2^k}\sum_{i=1}^{2^k} a_i + \frac{1}{2^k}\sum_{j=2^k+1}^{2^{k+1}} a_j = \frac{1}{2^k}\sum_{i=1}^{2^{k+1}} a_i.$$

Dividing by 2, we obtain the desired inequality for given $a_1, \cdots, a_{2^{k+1}}$, namely the inequality holds for $n = 2^{k+1}$. By the mathematical induction, given positive $a_1, \cdots, a_{2^m}$ with $n = 2^m$, the inequality holds.

Next, we assume that the inequality holds for $n = m \geq 2$. Then, given positive $a_1, \cdots, a_{m-1}$, we define

$$a_m = \frac{1}{m-1}\sum_{i=1}^{m-1} a_i.$$

Then,

$$\Big(a_m \prod_{i=1}^{m-1} a_i\Big)^{\frac{1}{m}} = \Big(\prod_{i=1}^{m} a_i\Big)^{\frac{1}{m}} \leq \frac{1}{m}\sum_{i=1}^{m} a_i = \frac{a_m}{m} + \frac{1}{m}\sum_{i=1}^{m-1} a_i = a_m.$$

Therefore,

$$\Big(\prod_{i=1}^{m-1} a_i\Big)^{\frac{1}{m}} \leq a_m^{\frac{m-1}{m}} = \Big(\frac{1}{m-1}\sum_{i=1}^{m-1} a_i\Big)^{\frac{m-1}{m}}.$$

Namely,

$$\Big(\prod_{i=1}^{m-1} a_i\Big)^{\frac{1}{m-1}} \leq \frac{1}{m-1}\sum_{i=1}^{m-1} a_i.$$

Since the inequality holds for $n = 2^m$, the mathematical induction guarantees the inequality for $n = 2^m - k$ for $k, m \in \mathbb{N}$ with $k < 2^m$. Hence, the inequality holds for every $n \in \mathbb{N}$. $\square$

Theorem 3 (Young's inequality for products)**.** *Given positive real numbers* x, y, p, q *with* $1 = \frac{1}{p} + \frac{1}{q}$*, the following holds*

$$xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q.$$

Proof. Given positive $x, y \in \mathbb{R}$ and $r, s \in \mathbb{Q}$ with $\frac{1}{r} + \frac{1}{s} = 1$, we can set $a = x^{\frac{1}{r}}, b = y^{\frac{1}{s}}$ and $r = 1 + \frac{m}{n} = \frac{n+m}{n}$ for some $n, m \in \mathbb{N}$. Then, $\frac{1}{s} = 1 - \frac{1}{r} = \frac{m}{n+m}$. The previous theorem yields

$$\begin{aligned}\frac{1}{n+m}\big(na + mb\big) =& \frac{1}{m+n}\big(a + \cdots + a + b + \cdots + b\big)\\ \geq& \Big(a \times \cdots a \times b \times \cdots \times b\Big)^{\frac{1}{n+m}} = a^{\frac{n}{n+m}} b^{\frac{m}{n+m}}.\end{aligned}$$

Namely,

$$\frac{1}{r}x^r + \frac{1}{s}y^s \geq xy.$$

Next, given real number $p > 1$, we choose a sequence of rational numbers $r_i > 1$ converging to p. Then, the sequence s_i defined by $\frac{1}{s_i} = 1 - \frac{1}{r_i}$ converges to q, where $\frac{1}{q} = 1 - \frac{1}{p}$. Thus, the limit location theorem implies

$$xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q = \lim \frac{1}{r_i}x^{r_i} + \frac{1}{s_i}y^{s_i}.$$

$\square$